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ON BRIESKORN'S THEOREM

BY

V. HINICH

Department of Theoretical Mathematics, The Weizmann Institute of Science, Rehovot 76100, Israel

ABSTRACT

A cohomological proof of Brieskorn's theorem describing the singularity of the nilpotent cone of a complex simple Lie algebra in a subregular point, is given.

1. Introduction

Let G be a complex simple algebraic group with the Lie algebra $\mathfrak{g}$. Let $\mathcal{N}$ denote the nilpotent cone of $\mathfrak{g}$.

It is well-known that $\mathcal{N}$ admits an open G-orbit $\mathcal{O}_{\text{reg}}$ (the regular nilpotent orbit of $\mathfrak{g}$) and that the complement $\mathcal{N} - \mathcal{O}_{\text{reg}}$ has codimension two in $\mathcal{N}$. An element $x \in \mathcal{N}$ is called **subregular** if it generates an orbit of codimension two in $\mathcal{N}$. The orbit of a subregular element is also called subregular. The singularity of $\mathcal{N}$ in a subregular point is essentially two-dimensional. The result of E. Brieskorn (cf. [B2]) claims that this singularity is a rational double point whose Dynkin graph coincides with that of the Lie algebra $\mathfrak{g}$ when the latter is of type A_n, D_n or E_6–E_8.

In his original proof [B2] Brieskorn calculates the equations which give singularities in subregular points and compares them with the known equations for rational double points. Slodowy in [Sl] reproves the result differently.

First of all, Slodowy proves that the nilpotent cone is normal in subregular points. (The result is due to Kostant, see [K], Th. 0.8.) After that, a cohomological calculation (apparently due to P. Deligne) shows that all self-intersection

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indices of the components of the exceptional divisor for a resolution of the singularity, are always equal to -2. Finally, the intersections of different components of the exceptional divisor are determined using a theorem of Tits (cf. [St], pp. 147–148).

In this note we calculate the intersection indices by purely cohomological methods. This gives a more conceptual explanation of the coincidence of Cartan matrices describing simple Lie algebras and their subregular singularities. Moreover, our calculation provides a proof of Tits' theorem *loc. cit.* which needs no case-by-case analysis. As a by-product we obtain that subregular orbits are unique (see [St], Th.1, p. 145).

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2. Desingularizations

It is pleasant to be able to start
with a nice desingularization
[St], p. 129

In (2.1) we describe a desingularization of the nilpotent cone $\mathcal{N}$ due to T. Springer, [Sp]. An appropriate base change (cf. (2.2)) gives a desingularization for the rational double point in question. All the constructions here are well-known (see [Sl]).

We shall equally use the notations ${}^g x = x^{g^{-1}}$ for the result of the adjoint action of $g \in G$ on $x \in \mathfrak{g}$

(2.1) Fix a Borel subgroup $B \subseteq G$, denote by $\mathfrak{b}$ the corresponding Borel subalgebra of $\mathfrak{g}$ and let $\mathfrak{n}$ be its nilradical. The tangent space to the flag variety G/B in a point gB identifies with $\mathfrak{g}\ /{}^g\mathfrak{b}$ and so the corresponding cotangent space identifies with ${}^g\mathfrak{n}$. Let $X = T^*(G/B)$ be the cotangent bundle of G/B. This can be considered as a sub-bundle of the trivial bundle $G/B \times \mathfrak{g}$ and so the composition

$$\pi : X \to G/B \times \mathfrak{g} \overset{pr_2}{\to} \mathfrak{g}$$

is defined. Its image coincides with $\bigcup {}^g\mathfrak{n} = \mathcal{N}$.

THEOREM ([Sp]): *The map $\pi : X \to \mathcal{N}$ constructed above is a desingularization.*

(2.2) Let now $x \in \mathcal{N}$ be a subregular element and let $T \subseteq \mathfrak{g}$ be a transverse slice for $\mathfrak{g}$ in x. Then $T \cap \mathcal{N}$ is two-dimensional having an isolated singularity in x and we choose T so that x is the only point of $T \cap \mathcal{N}$ not belonging to $\mathcal{O}_{\text{reg}}$. The scheme-theoretic inverse image $S = \pi^{-1}(T) \subseteq X$ is a desingularization for $T \cap \mathcal{N}$.

We are now able to formulate Brieskorn's theorem in a more general form proposed by P. Slodowy (cf. [Sl], 6.4).

THEOREM: *The surface $T \cap \mathcal{N}$ has a rational double point in x. If the Lie algebra $\mathfrak{g}$ has a homogeneous Dynkin graph then the singularity in x is described by the same graph. For the non-homogeneous case the correspondence is described by the table below.*

Type of $\mathfrak{g}$	Singularity
B_n	A_{2n-1}
C_n	D_{n+1}
F_4	E_6
G_2	D_4

3. Subregular Elements in $\mathfrak{n}$

We fix a nilpotent element $x \in \mathcal{N}$ and recall that the fixed point variety of x is defined as

$$\mathcal{B}_x = \{gB | x^g \in \mathfrak{n}\}.$$

Consider the following diagram

$$\begin{array}{l} G \xrightarrow{p} G/B \\ g \downarrow \\ G_x \end{array}$$

with $p(g) = gB, q(g) = x^g$.

One has $p^{-1}(\mathcal{B}_x) = q^{-1}({}^G x \cap \mathfrak{n})$. It is easy to see that qp^{-1} and pq^{-1} establish a one-to-one correspondence (**Spaltenstein correspondence**) between irreducible components of ${}^G x \cap \mathfrak{n}$ and $A_x(= \mathrm{Stab}_G(x)/\mathrm{Stab}^o_G(x))$-orbits on the set of irreducible components of $\mathcal{B}_x$ — see [Spa].

Let now x be subregular. Then the components of ${}^G x \cap \mathfrak{n}$ have dimension

$$\dim \mathcal{B}_x + \dim B - \dim\ \mathrm{Stab}_G x = \dim \mathfrak{n} + \dim \mathcal{B}_x - 2 = \dim \mathfrak{n} - 1.$$

These components are all B-stable by the construction and it is well-known that $\mathfrak{n}$ admits a dense B-orbit — this is the set $\mathcal{O}_{\mathrm{reg}} \cap \mathfrak{n}$ of regular elements in $\mathfrak{n}$. So the only candidates for the role of the components of ${}^G x \cap \mathfrak{n}$ are (open subsets of) the nilradicals $\mathfrak{m}_\alpha$ of the minimal parabolic subalgebras $\mathfrak{p}_\alpha$. On the other hand, Richardson's theorem [R], Prop.4, claims that ${}^G \mathfrak{m}_\alpha$ is a closed irreducible subset of codimension two in $\mathcal{N}$, and so it admits an open orbit of the same dimension. Thus all $\mathfrak{m}_\alpha$ appear eventually as the closures of components of ${}^G x \cap \mathfrak{n}$ with some subregular element x. A subregular orbit ${}^G x$ which is dense in a fixed $\mathfrak{m}_\alpha$ is obviously unique.

We shall need the following lemma which is a simple part of Tits' result [St], pp. 147–148.

LEMMA: (a) *Every irreducible component of $\mathcal{B}_x$ has a form gP_α/B for some $g \in G$ and a simple root α.*

(b) *Different components of $\mathcal{B}_x$ corresponding to the same simple root α do not meet; those corresponding to different simple roots either don't meet or meet transversally in one point.*

Proof: (a) Let C be a component of $\mathcal{B}_x$ and let $\mathfrak{m}_\alpha \cap {}^G x$ be the component of $\mathfrak{n} \cap {}^G x$ corresponding to C. Since $\mathfrak{m}_\alpha \cap {}^G x$ is P_α-invariant, its pre-image $q^{-1}(\mathfrak{m}_\alpha \cap {}^G x) \subseteq G$ is invariant under the right action of P_α. Thus the same is true for $C \subseteq G/B$ and, since C is one-dimensional, the only possibility for C is to be the image of gP_α.

(b) The first part of the claim immediately follows from (a). If two components $gP_\alpha/B, hP_\beta/B$ $(\alpha \neq \beta)$ meet in a point, one can suppose without loss of generality that $g = h = 1$ and that the intersection point is $1 \cdot B$. This is the only intersection point since $P_\alpha \cap P_\beta = B$ and the curves meet transversally since their tangent spaces are different.

The lemma is proven. ∎

4. Projective Lines on a Variety

(4.1) TYPE OF A LINE. Let X be a complex algebraic variety. We call a **projective line on** X a closed embedding $i : \mathbb{P}^1 \longrightarrow X$. The type of the projective line is defined as follows.

Consider the induced homorphism of cohomology algebras

$$i^* : H^\cdot(X,\mathbb{Z}) \longrightarrow H^\cdot(\mathbb{P}^1,\mathbb{Z}).$$

The cohomology of the projective line is well-known: one has $H^\cdot(\mathbb{P}^1,\mathbb{Z}) = \mathbb{Z}[x]/(x^2)$ where x has degree 2 and it is the class of the linear bundle $\mathcal{O}(1)$. So the ring homomorphism i^* defines (and is uniquely defined by) a $\mathbb{Z}$-homomorphism $H^2(X,\mathbb{Z}) \longrightarrow \mathbb{Z}$.

Definition: The **type** of $i : \mathbb{P}^1 \longrightarrow X$ is the element of $H^2(X,\mathbb{Z})^*$ defined above.

(4.2) INTERSECTION WITH DIVISORS. Let now X be non-singular and let D be a divisor on X. Suppose as in (4.1) we are given a curve $i : \mathbb{P}^1 \longrightarrow X$. The intersection index $(i, D)_X$ can be defined as the degree of the invertible sheaf $i^*(\mathcal{L}(D))$ on $\mathbb{P}^1$. In other words, if we denote by $\mathrm{cl}(D)$ the cohomology class of the divisor D (or, equivalently, of the invertible sheaf $\mathcal{L}(D)$) in $H^2(X,\mathbb{Z})$, then we have an obvious formula

$$(i, D)_X = \langle\ \text{type}\ (i),\ \mathrm{cl}\ (D)\rangle$$

where $\langle\ ,\ \rangle$ is the canonical pairing.

(4.3) A CALCULATION FOR THE FLAG VARIETY. Let now $X = G/B$ be a flag variety. Fix a simple root $\alpha \in \Delta$ and consider the line $i_\alpha : P_\alpha/B \longrightarrow G/B$. Recall that $H^2(G/B,\mathbb{Z}) = \mathfrak{h}^*_{\mathbb{Z}}$ in order that the class of a linear bundle $\mathcal{L}_\lambda$ corresponding to the character $e^\lambda : B \longrightarrow \mathbb{C}^*$ is exactly λ.

LEMMA: *The type of i_α is $-\alpha^\vee \in \mathfrak{h}_{\mathbb{Z}}$.*

Proof: The category of G-linear bundles over G/B is equivalent to the category of one-dimensional B-modules. The inverse image functor from the category of linear bundles over G/B to the category of linear bundles over $\mathbb{P}^1 = SL_2/B_2$ (B_2 being the corresponding Borel) is expressed in these terms as the forgetful functor from the category of B-modules to that of B_2-modules. In other words, the inverse image functor takes a character $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$ to its restriction $\lambda\ |_{\mathbb{C}\alpha^\vee}$. Recall that the line bundle over SL_2/B_2 corresponding to a character $\lambda : \mathbb{C}\alpha^\vee \longrightarrow \mathbb{C}$, is isomorphic to $\mathcal{O}(n)$ where $n = -\langle \alpha^\vee, \lambda\rangle$ which proves exactly what we need. ∎

5. Cartan Matrix in $H^\cdot(T^*(G/B))$

(5.1). We use here the notations of Sections 2–4. So, $S = \pi^{-1}(T)$ is a desingularization of $T \cap \mathcal{N}$ and $\pi^{-1}(x)$ is the corresponding exceptional divisor. The natural projection $\nu : X = T^*(G/B) \longrightarrow G/B$ identifies the exceptional divisor with the fixed point variety $\mathcal{B}_x$.

(5.2). Consider the following set of divisors D_α in $X = T^*(G/B)$ indexed by the simple roots of $\mathfrak{g}$ with respect to $\mathfrak{b}$:

$$D_\alpha = G \times^B \mathfrak{m}_\alpha \subseteq G \times^B \mathfrak{n} = T^*(G/B).$$

Note that $\mathcal{L}(D_\alpha) = \nu^*\mathcal{L}_\alpha$ and so the following result immediately follows from Lemma (4.3).

PROPOSITION: *Let C be a component of $\pi^{-1}(x)$ corresponding to a simple root α (see Section 3). Then $(C, D_\beta)_X = -\langle \alpha^\vee, \beta\rangle$.*

We wish to deduce from this the intersection matrix of the components of the exceptional divisor.

(5.3). The connection between the two Cartan matrices results from the following

LEMMA: *Let $j : S \longrightarrow X$ be the embedding map. Then $j^*(D_\alpha)$ is the sum of the components of $\pi^{-1}(x)$ corresponding to α.*

Proof: Note that (as a set of closed points) $\cup D_\alpha = \pi^{-1}(\mathcal{N} - \mathcal{O}_{\mathrm{reg}})$. Thus

$$(\cup D_\alpha) \cap S = \pi^{-1}((\mathcal{N} - \mathcal{O}_{\mathrm{reg}}) \cap T) = \pi^{-1}(x).$$

So

$$D_\alpha \cap S = D_\alpha \cap \pi^{-1}(x).$$

The image of the latter in G/B is exactly $\{gB \mid x^g \in \mathfrak{m}_\alpha\}$.

Thus it is clear at least that $j^*(D_\alpha) = \sum n_i C_i, n_i \geq 1$, the sum being taken over the components corresponding to α.

To prove that $n_i = 1$ we shall reproduce the calculation of [Sl] which shows that $(C,C)_S = -2$ for any component C. Here it is.

One knows that $(C,C)_S = \deg_C(N_{C/S})$ where $N_{C/S}$ denotes the normal bundle of C in S. Since $S = \pi^{-1}(T)$ one has by [EGA] IV 17.13.2 that $N_{S/X} = (\pi_{|S})^* N_{T/\mathfrak{g}}$ which is trivial for a sufficiently small T. So $(C,C)_S = \deg_C(N_{C/X})$. The embedding $i : C \longrightarrow X$ induces a short exact sequence

$$0 \longrightarrow T_C \longrightarrow i^* T_X \longrightarrow N_{C/X} \longrightarrow 0$$

T_C (resp. T_X) being the tangent bundle of C (resp. X). Since the tangent bundle of X is trivial, one obtains finally $(C,C)_S = -\deg(T_C) = -2$.

Finally, by Lemma 3 different components C, C' corresponding to the same simple root α do not meet. So, using Proposition (5.2) for a component C_i corresponding to the simple root α, one has $-2 = (C_i, D_\alpha)_X = n_i(C_i,C_i)_S = -2n_i$ so $n_i = 1$ as was claimed. ∎

6. Final

Proposition (5.2) and Lemma (5.3) determine the intersection matrix of the exceptional divisor. In fact, Lemma 3 asserts that any two different components have intersection index at most one and then by Section 5 any component of type $\alpha^\vee$ (note: the standard name for such a component is **the line of type** α!) meets exactly $\langle \alpha^\vee, \beta \rangle$ components of type $\beta^\vee$. Take any component of the exceptional divisor, say, of type $\alpha^\vee$. We see that for any simple root β not orthogonal to α the exceptional divisor contains also components of type $\beta^\vee$. Thus the exceptional divisor contains the components corresponding to all simple roots. This proves that the intersection matrix of the exceptional divisor for the desingularization $\pi : S \longrightarrow T \cap \mathcal{N}$ is exactly the Cartan matrix asserted by Theorem (2.2).

Remark: Another way of proving that any two different components have intersection index at most one was suggested by A. Joseph: if $\alpha \neq \beta$ then either $\langle \alpha^\vee, \beta \rangle$ or $\langle \beta^\vee, \alpha \rangle$ is not greater than 1 and so for components C of type $\alpha^\vee$ and C' of type $\beta^\vee$ one has $(C,C')_S \leq (C, D_\beta)_X = \langle \alpha^\vee, \beta \rangle \leq 1$ up to exchange of α and β. ∎

To accomplish the proof of Theorem (2.2) we proceed as in [Sl]: $T \cap \mathcal{N}$ is normal in x by *loc. cit.*, Lemma (2.2). Since the intersection matrix of the components of the exceptional divisor is Cartan matrix of type A, B or E, x is a rational double point of that type (cf. [B1]).

Note that we obtain as a by-product of the calculation made that there is a unique subregular orbit. In fact, the fiber $\pi^{-1}(x)$ in any subregular point $x \in \mathfrak{g}$ contains the components of all possible types. This means that the orbit ${}^G x$ is dense in $\mathfrak{m}_\alpha$ for any simple root α. Thus any two subregular orbits should coincide.

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